Exotic galilean symmetry, non-commutativity & the Hall effect*

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Abstract

The "exotic" particle model associated with the two-parameter central extension of the planar Galilei group can be used to derive the ground states of the Fractional Quantum Hall Effect. Similar equations arise for a semiclassical Bloch electron. Exotic Galilean symmetry can also be shared by Chern-Simons field theory of the Moyal type.

1 Introduction

Recent interest in non-commuting structures stems, as it often happens, from far remote fields. In high-energy physics, it comes from the theory of strings and membranes [1], or from studying galilean symmetry in the plane [2, 3, 4, 5]. Independently and around the same time, very similar structures were considered in condensed matter physics, namely for the semiclassical dynamics of a Bloch electron [6]. Recent developments include the Anomalous [7], the Spin [8] and the Optical [9] Hall effects.

Below we first review the exotic point-particle model of [4], followed by a brief outline of the semiclassical Bloch electron [6].

Our present understanding of the Fractional Quantum Hall Effect is based on the motion of charged vortices in a magnetic field [10, 11]. Such vortices arise as exact solutions in a field theory of matter coupled to an abelian gauge field A_ν, whose dynamics is governed by the Chern-Simons term [12, 13]. Such theories can be either relativistic or nonrelativistic. In the latter case, boosts commute. Exotic Galilean symmetry can be found, however, in a Moyal-version of Chern-Simons field-theory [5], presented in Section 4.

2 "Exotic" mechanics in the plane

It has been known for (at least) 33 years that the planar Galilei group admits an "exotic" two-parameter central extension [3]: unlike in $D \geq 3$ spatial dimensions, the commutator of galilean boosts yields a new central charge,

$$[\mathcal{G}_1, \mathcal{G}_2] = \kappa. \tag{2.1}$$

This has long remained a sort of mathematical curiosity, though. It has been around 1995 that people started to inquire about the physical consequences of such an extended symmetry.

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In [2, 4], in particular, Souriau's "orbit method" [14] was used to construct a classical system with such an exotic symmetry. The latter is realized by the usual galilean generators, except for the boost and the angular momentum,

$$
\begin{aligned}
j &= \epsilon_{ij}x_i p_j + \tfrac{1}{2}\theta\, p_i p^i, \\
\mathcal{G}_i &= m x_i - p_i t + m\theta\,\epsilon_{ij}p_j.
\end{aligned}
\tag{2.2}
$$

The resulting free model moves, however, exactly as in the standard case. The "exotic" structure behaves hence roughly as spin: it contributes to some conserved quantities, but the new terms are separately conserved. The new structure does not seem to lead to any new physics.

The situation changes dramatically if the particle is coupled to a gauge field. The resulting equations of motion read

$$
\begin{aligned}
m^*\dot{x}_i &= p_i - em\theta\,\varepsilon_{ij}E_j, \\
\dot{p}_i &= eE_i + eB\,\varepsilon_{ij}\dot{x}_j,
\end{aligned}
\tag{2.3}
$$

where $\theta = k/m^2$ is the non-commutative parameter and we have introduced the *effective mass*

$$
m^* = m(1 - e\theta B). \tag{2.4}
$$

The changes, crucial for physical applications, are two-fold: Firstly, the relation between velocity and momentum, (3.1), contains an "anomalous" term so that $\dot{x}_i$ and p_i are not parallel. The second novelty is the interplay between the exotic structure and the magnetic field, yielding the effective mass m^* in (3.2).

The equations (2.3) come from the Lagrangian

$$
\int (\mathbf{p} - \mathbf{A}) \cdot d\mathbf{x} - \frac{p^2}{2}\,dt + \frac{\theta}{2}\,\mathbf{p} \times d\mathbf{p}. \tag{2.5}
$$

When $m^* \neq 0$, 2.3 is also a Hamiltonian system, $\dot{\xi} = \{h, \xi^\alpha\}$, with $\xi = (p_i, x^j)$ and Poisson brackets

$$
\begin{aligned}
\{x_1, x_2\} &= \frac{m}{m^*}\,\theta, \\
\{x_i, p_j\} &= \frac{m}{m^*}\,\delta_{ij}, \\
\{p_1, p_2\} &= \frac{m}{m^*}\,eB,
\end{aligned}
\tag{2.6}
$$

A most remarkable property is that for vanishing effective mass $m^* = 0$ i.e. when the magnetic field takes the critical value

$$
B = \frac{1}{e\theta}, \tag{2.7}
$$

then the system becomes singular. Then "Faddeev-Jackiw" (alias symplectic) reduction yields an essentially two-dimensional, simple system, similar to "Chern-Simons mechanics" [15]. The symplectic plane plays, simultaneously, the role of both configuration and phase space. The only motions are those which follow a generalized Hall law, and the quantization of the reduced system yields the "Laughlin wave functions" [10], which are in fact the ground states in the Fractional Quantum Hall Effect (FQHE).

The relations (2.6) diverge as $m^* \to 0$, but after reduction we have, cf. (2.1),

$$
\{x_1, x_2\} = \theta. \tag{2.8}
$$

3 Semiclassical Bloch electron

Quite remarkably, around the same time and with no relation to the above developments, a very similar theory has arisen in solid state physics [6]. Applying a Berry-phase argument to a Bloch electron in a lattice, a semiclassical model can be derived. The equations of motion in the n^{th} band read

$$\dot{\mathbf{x}} = \frac{\partial\epsilon_n(\mathbf{p})}{\partial\mathbf{p}} - \dot{\mathbf{p}}\times\vec{\Omega}(\mathbf{p}), \tag{3.1}$$

$$\dot{\mathbf{p}} = -e\mathbf{E} - e\dot{\mathbf{x}}\times\mathbf{B}(\mathbf{x}), \tag{3.2}$$

where $\mathbf{x} = (x^i)$ and $\mathbf{p} = (p_j)$ denote the electron's three-dimensional intracell position and quasimomentum, respectively, $\epsilon_n(\mathbf{p})$ is the band energy. The purely momentum-dependent $\vec{\Omega}$ is the Berry curvature of the electronic Bloch states, $\Omega_i(\mathbf{p}) = \epsilon_{ijl}\partial_{\mathbf{p}_j}\mathcal{A}_l(\mathbf{p})$, where $\mathcal{A}$ is the Berry connection.

Recent applications of the model, based on the anomalous velocity term in (3.1), include the Anomalous [7] and the Spin [8] Hall Effects.

Eqns. (3.1-3.2) derive from the Lagrangian

$$L^{Bloch} = (p_i - eA_i(\mathbf{x},t))\dot{x}^i - (\epsilon_n(\mathbf{p}) - eV(\mathbf{x},t)) + \mathcal{A}^i(\mathbf{p})\dot{p}_i, \tag{3.3}$$

and are also consistent with the Hamiltonian structure [17, 16]

$$\{x^i, x^j\}^{Bloch} = \frac{\varepsilon^{ijk}\Omega_k}{1+e\mathbf{B}\cdot\vec{\Omega}}, \tag{3.4}$$

$$\{x^i, p_j\}^{Bloch} = \frac{\delta^i{}_j + eB^i\Omega_j}{1+e\mathbf{B}\cdot\vec{\Omega}}, \tag{3.5}$$

$$\{p_i, p_j\}^{Bloch} = -\frac{\varepsilon_{ijk}eB^k}{1+e\mathbf{B}\cdot\vec{\Omega}} \tag{3.6}$$

and Hamiltonian $h = \epsilon_n - eV$.

Restricted to the plane, these equations reduce, furthermore, to the exotic equations (2.3) provided $\Omega_i = \theta\delta_{i3}$. For $\epsilon_n(\mathbf{p}) = \mathbf{p}^2/2m$ and chosing $\mathcal{A}_i = -(\theta/2)\epsilon_{ij}p_j$, the semiclassical Bloch Lagrangian (3.3) becomes the "exotic" expression (2.5). The exotic galilean symmetry is lost, however, if θ is not constant.

4 Non-commutative Chern-Simons theory

Field theory coupled to an abelian gauge field A_ν, whose dynamics is governed by the Chern-Simons term admits exact vortex solutions [12, 13]. Such theories can be either relativistic or nonrelativistic. In the latter case [13],

$$L = L_{matter} + L_{field} = i\bar{\psi}D_t\psi - \tfrac{1}{2}|\mathbf{D}\psi|^2 + \mu\left(\tfrac{1}{2}\epsilon_{ij}\partial_t A_i A_j + A_t B\right), \tag{4.1}$$

[plus a potential $U(\psi)$], where $D_\nu = \partial_\nu - ieA_\nu$, $\nu = t, i$. Infinitesimal galilean boosts, implemented conventionally as

$$\delta^0\psi = i\mathbf{b}\cdot\mathbf{x}\,\psi - t\mathbf{b}\cdot\vec{\nabla}\psi, \tag{4.2}$$

$$\delta^0 A_i = -t\mathbf{b}\cdot\vec{\nabla}A_i, \tag{4.3}$$

$$\delta^0 A_t = -\mathbf{b}\cdot\mathbf{A} - t\mathbf{b}\cdot\vec{\nabla}A_t, \tag{4.4}$$

are generated by the constants of the motion

$$\mathcal{G}_i^0 = t\mathcal{P}_i - \int x_i\,|\psi|^2\,d^2\mathbf{x}, \qquad \mathcal{P}_i = \int \frac{1}{2i}(\bar{\psi}\partial_i\psi - (\overline{\partial_i\psi})\psi)d^2\mathbf{x} - \frac{\mu}{2}\int \epsilon_{jk}A_k\partial_i A_j d^2\mathbf{x}. \tag{4.5}$$

The galilean symmetry extends in fact into a Schrödinger symmetry [13]; there is no sign of "exotic" galilean symmetry, however, since $\{\mathcal{G}_1^0, \mathcal{G}_2^0\} = 0$. Replacing ordinary products with the Moyal star-product,

$$(f \star g)(x_1, x_2) = \exp\left(i\frac{\theta}{2}(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1})\right) f(x_1, x_2)g(y_1, y_2)\Big|_{\mathbf{x}=\vec{y}} \tag{4.6}$$

where θ is a real parameter, a non-commutative version of the theory can be constructed, though. The classical Lagrangian is formally still (4.1), but the covariant derivative, the field strength, and the Chern-Simons term,

$$D_\mu\psi \;=\; \partial_\mu - ieA_\mu \star \psi, \tag{4.7}$$

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu - ie(A_\mu \star A_\nu - A_\nu \star A_\mu), \tag{4.8}$$

$$\text{Chern-Simons term} \;=\; \frac{\mu}{2}\,\epsilon_{\mu\nu\sigma}\left(A_\mu \star \partial_\nu A_\sigma - \frac{2ie}{3}A_\mu \star A_\nu \star A_\sigma\right), \tag{4.9}$$

respectively, all involve the Moyal form. The variational equations read

$$iD_t\psi + \frac{1}{2}\mathbf{D}^2\psi \;=\; 0, \tag{4.10}$$

$$\kappa E_i - e\epsilon_{ik}\jmath^l{}_k \;=\; 0, \tag{4.11}$$

$$\kappa B + e\rho^l \;=\; 0, \tag{4.12}$$

where $B = \epsilon_{ij}F_{ij}$, $E_i = F_{i0}$, and ρ^l and $\jmath^{\,l}$ denote the *left density* and *left current*, respectively,

$$\rho^l = \psi \star \bar{\psi}, \qquad \jmath^{\,l} = \frac{1}{2i}\left(\mathbf{D}\psi \star \bar{\psi} - \psi \star (\overline{\mathbf{D}\psi})\right). \tag{4.13}$$

These equations admit, just like their ordinary counterparts, exact vortex solutions [18].

The modified theory is *not* invariant w. r. t. boosts implemented as above. Galilean invariance can be restored, however, by implementing boosts rather as

$$\delta\psi = \psi \star (i\mathbf{b}\cdot\mathbf{x}) - t\mathbf{b}\cdot\vec{\nabla}\psi = (i\mathbf{b}\cdot\mathbf{x})\psi + \frac{\theta}{2}\mathbf{b}\times\vec{\nabla}\psi - t\mathbf{b}\cdot\vec{\nabla}\psi, \tag{4.14}$$

supplemented by (4.3)-(4.4). Then the generators,

$$\mathcal{G}_i = t\mathcal{P}_i - \int x_i\bar{\psi}\star\psi\,d^2\mathbf{x}, \tag{4.15}$$

do satisfy the "exotic" relation (2.1)

$$[\mathcal{G}_1, \mathcal{G}_2] = \kappa \qquad \text{with} \qquad \kappa = -\theta\int|\psi|\,d^2\mathbf{x}. \tag{4.16}$$

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References

[1] N. Seiberg and E. Witten, *JHEP* **9909** 032 (1999).

[2] C. Duval, *Exotic Galilei group, IQHE and Chern-Simons electrodynamics.* Unpublished draft (1995); D. R. Grigore, *Journ. Math. Phys.* **37**, 240 (1996); *ibid.* **37** 460 (1996);

[3] Y. Brihaye, C. Gonera, S. Giller and P. Kosiński, `hep-th/9503046` (unpublished); J. Lukierski, P. C. Stichel, W. J. Zakrzewski, *Annals of Physics* (N. Y.) **260**, 224 (1997). The "exotic" two-parameter extension of the planar Galilei goup has been first noticed by J.-M. Lévy-Leblond, in *Group Theory and Applications* (Loebl Ed.), **II**, Acad. Press, New York, p. 222 (1972). See also A. Ballesteros, M. Gadella, and M. del Olmo, *Journ. Math. Phys.* **33**, 3379 (1992).

[4] C. Duval and P. A. Horváthy, *Phys. Lett.* **B 479**, 284 (2000) [`hep-th/0002233`] C. Duval and P. A. Horváthy, *Journ. Phys.* **A 34**, 10097 (2001) [`hep-th/0106089`]; P. A. Horváthy, *Ann. Phys.* **299**, 128 (2002) [`hep-th/0201007`].

[5] P. A. Horváthy, L. Martina and P. Stichel, *Nucl. Phys.* **B 673**, 301-318 (2003).

[6] M. C. Chang and Q. Niu, *Phys. Rev. Lett.* **75**, 1348 (1995); *Phys. Rev.* **B 53**, 7010 (1996); G. Sundaram and Q. Niu, *Phys. Rev.* **B59**, 14915 (1999). See also A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu and J. Zwanziger, *The Geometric Phase in Quantum Systems.* Chapter 12. Springer Verlag (2003).

[7] T. Jungwirth, Q. Niu, and A. H. MacDonald, *Phys. Rev. Lett.* **90**, 207208 (2002); D. Culcer, A. H. MacDonald, and Q. Niu, *Phys. Rev.* **B 68**, 045327 (2003); Fang et al. *Science* **302**, 92 (2003); Similar theories were initiated by R. Karplus and J. M. Luttinger, *Phys. Rev.* **95**, 1154 (1954).

[8] S. Murakami, N. Nagaosa, and S.-C. Zhang, *Science* **301**, 1348 (2003) [`cond-mat/0308167`]; Y.-S. Wu, talk given at this Meeting.

[9] K. Yu. Bliokh and Yu. P. Bliokh, *Phys. Lett.* **A333**, 181 (2004) [`physics/0402110`]; M. Onoda, S. Murakami, N. Nagaosa, *Phys. Rev. Lett.* **93**, 083901 (2004).

[10] R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983). On the Quantum Hall Effect see, e.g., *Quantum Hall Effect*, Ed. M. Stone, World Scientific Singapore (1992).

[11] S.-C. Zhang, *Intl. Journ. Mod. Phys.* **6**, 25 (1992).

[12] H. J. De Vega and F. A. Schaposnik, *Phys. Rev. Lett.* **56**, 2564 (1986); S. K. Paul and A. Khare, *Phys. Lett.* **B174**, 420 (1986); S.-C. Zhang, T. H. Hansson and S. Kivelson, *Phys. Rev. Lett.* **62**, 82 (1989); J. Hong, Y. Kim and P. Y. Pac, *Phys. Rev. Lett.* **64**, 2230 (1990); R. Jackiw and E. Weinberg, *Phys. Rev. Lett.* **64**, 2234 (1990).

[13] R. Jackiw and S. Y. Pi, *Phys. Rev. Lett.* **64**, 2969 (1990); *Phys. Rev.* **D42**, 3500 (1990). For a review see, e. g., R. Jackiw and S. Y. Pi, *Self-dual Chern-Simons Solitons.* Prog. Theor. Phys. Suppl. **107**, 1 (1992), or G. Dunne, *Self-Dual Chern-Simons Theories.* Springer Lecture Notes in Physics. New Series: Monograph 36. (1995). See also C. Duval and P. A.

Horváthy, *Chern-Simons vortices*. In Proc. Clifford Conf., Madeira'95, H. Nencka and J.-P. Bourguignon (eds). *Contemporary Mathematics*, **203**, p. 271 (1997) [`hep-th/0307025`].

[14] J.-M. Souriau, *Structure des systèmes dynamiques*, Dunod: Paris (1970); *Structure of Dynamical Systems: a Symplectic View of Physics*. Birkhäuser: Dordrecht (1997).

[15] G. Dunne R. Jackiw and C. A. Trugenberger, *Phys. Rev.* **D41**, 661 (1990); G. Dunne and R. Jackiw, *Nucl. Phys.* **B** (Proc. Suppl.) **33C** 114 (1993). The model goes back to G. Kirchhoff, *Vorlesungen über mathematischen Physik. Mechanik*. 3rd. ed., pp. 251-272. G. B. Teubner, Leipzig (1883), see P. Horváthy, talk given at the *COSLAB-VORTEX-BEC 2000+ Workshop*. Bilbao'03. [`hep-th/0307175`].

[16] P. A. Horváthy, L. Martina and P. Stichel, *Phys. Lett.* **B 615**, 87 (2005). [`hep-th/0412090`]. C. Duval, Z. Horváth, P. A. Horváthy, L. Martina and P. Stichel, to be published in *Mod. Phys. Lett.* **B**. [`cond-mat/0506051`].

[17] A. Bérard and H. Mohrbach, *Phys. Rev.* **D 69**, 127701 (2004) [`hep-th/0310167`].

[18] G. S. Lozano, E. F. Moreno, F. A. Schaposnik, *Journ. High Energy Phys.* **02** 036 (2001) [`hep-th/0012266`]; D. Bak, S. K. Kim, K.-S. Soh, and J. H. Yee, *Phys. Rev.* **D64**, 025018 (2001) [`hep-th/0102137`].